LEBANESE AMERICAN UNIVERSITY

Electrical and Computer Engineering Dept

COE 755

Queueing Theory

Spring 2013
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Project III

Due date: Tuesday May 14 2013

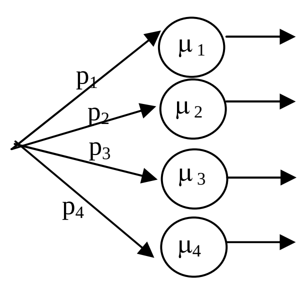
Description

Let us consider a real-life single server, queue for which we have only a sample of service times. We want to model this service time using a Coxian 2 model. To do this, we will first fit a C_2 distribution to the sample of observations. The following steps will be carried out:

1. Create a sample of service times using simulation techniques
2. Calculate the first three sample moments
3. Fit a C_2 using the method of moments

Detailed Instructions

Step 1:
The sample of service times will be created by generating stochastic variates from the following four-phased hyper-exponential distribution:



Where p_1=0.2, p_2=0.3, p_3=0.3, p_4=0.2, and μ_1=1, μ_2=2, μ_3=3, μ_4=40. Use the following procedure to generate stochastic variates:

1. Sample a pseudo-random number r.
2. Locate the interval within which r falls in order to determine from which exponential distribution to use:
   - If $r \le p_1$ then draw a variate from $\exp(\mu_1)$
   - If $p_1 < r \le p_1 + p_2$ then draw a variate from $\exp(\mu_2)$
   - If $p_1 + p_2 < r \le p_1 + p_2 + p_3$ then draw a variate from $\exp(\mu_3)$
   - Otherwise, draw a variate from $\exp(\mu_4)$

Step 2:
Obtain a sample of 2,100 observations, and calculate the following sample moments after you ignore the first 100 observations:

$$\overline{X} = \frac{1}{2000}\sum_{i=1}^{2000} x_i, \quad \overline{X^2} = \frac{1}{2000}\sum_{i=1}^{2000} x_i^2, \quad \overline{X^3} = \frac{1}{2000}\sum_{i=1}^{2000} x_i^3$$

Step 3:
Fit in a C_2 using the method of moments, summarized at the end of this assignment, by setting: $m_1 = \overline{X}$, $m_2 = \overline{X^2}$, $m_3 = \overline{X^3}$

Step 4:
Construct a frequency histogram of the sample service times. (Divide the x-axis into intervals of the same length and observe the number of times the sample service time falls within each interval. Divide each of these numbers by the total number of observations. This will give you the frequency with which the service time falls within each interval. This frequency is indicated on the y axis).

Now, calculate the area under the pdf of the C_2 for each interval of the histogram and plot the results against the histogram. Visually observe the accuracy of the fitted C_2 distribution. The pdf of the C_2 distribution is given by the expression:

$$f(x) = b\mu_1 e^{-\mu_1 x} + a[\frac{\mu_1\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 x} + \frac{\mu_1\mu_2}{\mu_1 - \mu_2} e^{-\mu_2 x}]$$

(Instead of integrating under the curve of the pdf, calculate the value of the pdf of the C_2 at the mid-point of each interval of the histogram and then multiply by the interval's width. This will give you a good approximation if the width of the interval is small).

Method of moments

Consider a C_2 with parameters μ_1, μ_2 and a (1-a=b). Then the first three moments are:

$$E(X) = \frac{1}{\mu_1} + \frac{a}{\mu_2}$$

$$E(X^2) = \frac{2b}{\mu_1^2} - \frac{2a\mu_1\mu_2 - 2a(\mu_1 + \mu_2)^2}{\mu_1^2\mu_2^2}$$

$$E(X^3) = \frac{6b}{\mu_1^3} - \frac{12a\mu_1\mu_2(\mu_1 + \mu_2) - 6a(\mu_1 + \mu_2)^3}{\mu_1^3\mu_2^3}$$

Let m_1, m_2, m_3 be the first three moments of the distribution which we want to approximate by a C_2. Then, by equating m_1=E(X), m_2=E(X^2), and m_3 = E(X^3), we have:

$$a = \frac{\mu_2}{\mu_1}(m_1\mu_1 - 1)$$

$$\mu_1 = \frac{X + \sqrt{X^2 - 4Y}}{2}$$

$$\mu_2 = X - \mu_1$$

where

$$Y = \frac{6m_1 - 3m_2/m_1}{(6m_2^2/4m_1) - m_3} \text{ and } X = \frac{1}{m_1} + \frac{m_2 Y}{2m_1}.$$

The following condition has to hold: $3m_2^2 < 2m_1m_3$. Note that the above method of moment applies to the case where the c^2 of the original distribution (which we approximate by a C_2) is greater than 1.

If this condition does not hold, then we can use the nearest acceptable value of the third moment, or do the following two-moment fit:

$$\mu_1=2/m_1,\ \mu_2=1/m_1c^2 \text{ and } a=1/2c^2$$

where m_1 and c^2 are given. These expressions can be used when the c^2 of the original distribution (which we approximate by a C_2) is greater than 0.5.