LEBANESE AMERICAN UNIVERSITY
Electrical and Computer Engineering Dept

COE 555

Queueing Theory

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Exam Preparation Problems
(Miscellaneous)

Problem I (Poisson distribution)

Raindrops are falling at an average rate of 20 drops per square inch per minute. What would be a reasonable distribution to use for the number of raindrops hitting a particular region measuring 5 inches^2 in t minutes? Why? Using your chosen distribution, compute the probability that the region has no rain drops in a given 3 second time interval.

Solution

A reasonable choice of distribution is Poisson(λt), where λ= 20·5 = 100 (the average number of raindrops per minute hitting the region). Assuming this distribution,

$$P\left(no\ raindrops\ in\ \frac{1}{20}\ of\ a\ minute\right) = \frac{e^{-\frac{100}{20}}\left(\frac{100}{20}\right)^0}{0!} = e^{-5}$$

Problem II (LOTUS)

Consider a discrete random variable X that follows a Poisson distribution with a rate parameter λ. Find E(X!) (the average of the factorial of X), if it is finite.

Solution

By LOTUS,

$$E(X!) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{k!\,\lambda^k}{k!} = \frac{e^{-\lambda}}{1-\lambda}$$

For 0<λ<1 since this is a geometric series and (E(X!) is infinite if λ≥1)

Problem III (LOTUS)

Find E(X^3) given that X is a random variable that follows the exponential distribution with a rate parameter λ.

Solution

By LOTUS, we have:

$$E(X^3) = \int_0^{\infty} x^3\lambda e^{-\lambda x}dx = \frac{6}{\lambda^3}$$

Problem IV (2D-LOTUS)

Let X and Y be i.i.d. Unif(0, 1). Find the expected value and the standard deviation of the distance between X and Y.

Solution

Let W = |X-Y|. By 2D-LOTUS, we have:

$$E(W) = \int_0^1 \int_0^1 |x-y| dxdy = \frac{1}{3}$$

Next, we find $E(W^2)$. This can either be done by computing the following double integral

$$E(W^2) = \int_0^1 (x-y)^2 dxdy$$

Or by writing

$$E(W^2) = E[(X-Y)^2] = E(X^2) + E(Y^2) - 2E(XY)$$

Which is

$$2E(X^2) - 2(EX)^2 = 2Var(X) = \frac{1}{6}$$

Since E(XY) = E(X)E(Y) for X, Y independent and E(X) = E(Y) and $E(X^2) = E(Y^2)$ (as X and Y have the same distribution). Thus, E(W) = 1/3.

$Var(W) = E(W^2) - (E(W))^2 = 1/18$.

As such, the standard deviation of the distance between X and Y is $\frac{1}{\sqrt{18}} = \frac{1}{3\sqrt{2}}$

Problem V (Correlation)

A chicken lays a Poisson(λ) number N of eggs. Each egg, independently, hatches a chick with probability p. Let X be the number that hatch, so X|N=n is Binomial(n, p). Find the correlation between N and X.

Solution

As shown in class, in this story X and Y (the number of eggs that don't hatch) are independent, with X following a Poisson(λp) and Y following a Poisson(λq), for q = 1 – p. So,

$$Cov(N,X) = Cov(X+Y,X) = Cov(X,X) + Cov(Y,X) = Var(X) = \lambda p$$

This gives

$$Correlation(N,X) = Corr(N,X) = \frac{Cov(N,X)}{SD(N)SD(X)} = \frac{\lambda p}{\sqrt{(\lambda \lambda p)}} = \sqrt{p}$$